Splitting the differential Riccati equation

Tony Stillfjord

Numerical Analysis, Lund University

Joint work with Eskil Hansen

Innsbruck
Okt 15, 2014
Outline

Splitting methods for evolution equations

The differential Riccati equation

Implementation issues and results
Outline

Splitting methods for evolution equations

The differential Riccati equation

Implementation issues and results
Abstract evolution equations

We consider

\[ \dot{u} = (F + G) u, \quad u(0) = u_0 \]

on a Hilbert or Banach space \( H \)

\( F \) typically (nonlinear) diffusion operator

\( G \) (nonlinear) perturbation with sufficiently good properties
Splitting methods

Full problem

\[ \dot{u} = (F + G)u, \quad u(0) = u_0 \]

difficult and/or expensive

Instead cleverly combine solutions to sub-problems

\[ \dot{u} = Fu \quad \text{and} \quad \dot{u} = Gu \]

Simple and/or cheaper
Splitting methods: examples

\[ \mathcal{L}_h = e^{hF} e^{hG} \quad \text{Lie} \]

\[ S_h = e^{h/2G} e^{hF} e^{h/2G} \quad \text{Strang} \]

\[ \mathcal{M}_h = (I - hF)^{-1} e^{hG} \quad \text{Mixed Lie} \]

Solution approximations to \( u(nh) \) are

\[ \mathcal{L}_h^n u_0, \quad S_h^n u_0, \quad \mathcal{M}_h^n u_0 \]
Dissipative setting

Dissipative problems (Hilbert space)

\[(Fu - Fv, u - v)_H \leq 0 \quad u, v \in \mathcal{D}(F)\]

Further

\[\mathcal{D}(F) \subset C, \quad C \text{ closed, convex}\]

with range condition

\[C \subset \mathcal{R}(I - hF), \quad h \geq 0\]
Consequences

Non-expansive resolvent

\[ \| (I - hF)^{-1}u - (I - hF)^{-1}v \|_H \leq \|u - v\|_H \]

⇒ will give stability for the schemes

Semigroup

\[ e^{tF}u_0 = \lim_{n \to \infty} (I - hF)^{-n}u_0, \quad t = nh \]

is mild solution to \[ \dot{u} = Fu \quad \text{if} \quad u_0 \in \overline{D(F)} \]

⇒ existence of solutions, but no further regularity
Splitting analysis

Convergence of various splitting schemes:

Brézis & Pazy 1972, Barbu 1976, etc.

But how fast? Convergence orders

Basic result, Crandall & Liggett 1971:

$$\| (I - hF)^{-n} u_0 - u(nh) \|_H \leq C h^p \quad p \in [1/2, 1]$$

Note: implicit Euler discretization for $\dot{u} = Fu$!
Analysis idea

**Stability** from dissipativity

**Consistency** by estimating distance to implicit Euler, $O(h^q)$

**Convergence** of order $O(h^p + h^q)$
Outline

Splitting methods for evolution equations

The differential Riccati equation

Implementation issues and results
The differential Riccati equation (DRE)

\[
\dot{P}(t) = A^* \circ P(t) + P(t) \circ A + Q - P(t) \circ P(t),
\]

\[P(0) = P_0\]

Solve for operator-valued \( P(t) \) for \( t \in [0, T] \)

Assumption: \( Q \) and \( P_0 \) self-adjoint, positive semi-definite

Important in e.g. optimal control - LQR problems
DRE splitting

\[
\dot{P}(t) = A^* \circ P(t) + P(t) \circ A + Q - P(t) \circ P(t)
\]

\[
= \mathcal{F} P + \mathcal{G} P
\]

Mixed Lie splitting scheme:

\[
\mathcal{M}_h = (I - h\mathcal{F})^{-1} e^{h\mathcal{G}}
\]
More problems, but simpler

Nonlinear subproblem:

\[ e^{hG} P_0 = (I + hP_0)^{-1} P_0 \]

Affine subproblem:

\[ U = (I - hF)^{-1} P_0 \]

equivalent to

\[ (I - 2hA)^* U + U(I - 2hA) = 2hQ + 2P_0 \]

(Linear) Lyapunov equation for \( U \)
The dissipative setting: Which space?

Given Gelfand triple

\[ V \hookleftarrow H \cong H^* \hookrightarrow V^* \]

\( A, A^* \in \mathcal{L}(V, V^*) \) given by

\[ \langle -Au, v \rangle_{V^* \times V} = a(u, v) \]

\[ \langle -A^* u, v \rangle_{V^* \times V} = a(v, u) \]

with bounded, coercive \( a \):

\[ |a(u, v)| \leq C_1 \|u\|_V \|v\|_V \quad \text{and} \quad a(u, u) \geq C_2 \|u\|^2_V \]
The dissipative setting: Which space?

Given Gelfand triple

\[ V := H^1_0(\Omega) \hookrightarrow L^2(\Omega) \cong L^2(\Omega)^* \hookrightarrow H^{-1}(\Omega) =: V^* \]

\[ A, A^* \in \mathcal{L}(H^1_0, H^{-1}) \text{ given by} \]

\[ \langle -Au, v \rangle_{V^* \times V} = (\sqrt{\alpha} \nabla u, \sqrt{\alpha} \nabla v)_{L^2} + \lambda (u, v)_{L^2} \]

\[ = \langle -A^* u, v \rangle_{V^* \times V} \]

Diffusion operator

\[ A = \nabla \cdot (\alpha \nabla u) - \lambda I \]
We look for $P(t)$ in

$$\mathcal{H} = \mathcal{HS}(H, H)$$

Stronger than bounded linear operator $\mathcal{L}(H, H)$

Hilbert space with

$$\langle f, g \rangle_\mathcal{H} = \sum_{k=1}^{\infty} \langle fe_k, ge_k \rangle_\mathcal{H}$$
Why Hilbert-Schmidt?

New Gelfand triple

\[ \mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^* \]

The operators $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ with

\[ \mathcal{F} P = A^* \circ P + P \circ A + Q \]
\[ \mathcal{G} P = -P \circ P \]

\[ \mathcal{D}(\mathcal{F}) = \{ P \in \mathcal{V} ; \mathcal{F} P \in \mathcal{H} \} \]
\[ \mathcal{D}(\mathcal{G}) = \mathcal{C} \]

where

\[ \mathcal{C} = \{ P \in \mathcal{H} ; P = P^* ; (Px, x) \geq 0 \} \]
Why Hilbert-Schmidt?

New Gelfand triple

\[ \mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^* \]

The operators \( \mathcal{F}, \mathcal{G} \) and \( \mathcal{F} + \mathcal{G} \) with

\[
\mathcal{F} P = A^* \circ P + P \circ A + Q \quad \quad \mathcal{G} P = -P \circ P
\]

\[
D(\mathcal{F}) = \{P \in \mathcal{V} ; \mathcal{F} P \in \mathcal{H}\} \quad \quad D(\mathcal{G}) = \mathcal{C}
\]

where

\[
\mathcal{C} = \{P \in \mathcal{H} ; P = P^* ; (Px, x) \geq 0\}
\]

are all dissipative:

\[
(\mathcal{F} P_1 - \mathcal{F} P_2, P_1 - P_2)_{\mathcal{H}} \leq 0
\]
Why Hilbert-Schmidt?

New Gelfand triple

\[ V \hookrightarrow H \cong H^* \hookrightarrow V^* \]

The operators \( F, G \) and \( F + G \) with

\[ FP = A^* \circ P + P \circ A + Q \quad \quad GP = -P \circ P \]

\[ \mathcal{D}(F) = \{ P \in V \mid FP \in H \} \quad \quad \mathcal{D}(G) = \mathcal{C} \]

where

\[ \mathcal{C} = \{ P \in H \mid P = P^* ; (Px, x) \geq 0 \} \]

all satisfy the range condition:

\[ (I - hF)^{-1}C \subset C \]
Convergence order

Mixed Lie

$$\mathcal{M}_h = (I - h\mathcal{F})^{-1}e^{h\mathcal{G}}$$

Stability by dissipativity of $\mathcal{F}$, $\mathcal{G}$

Consistency for $P_0 \in \mathcal{D}(\mathcal{F}) \cap C$

$$\|\mathcal{M}_h^n P_0 - (I - h(\mathcal{F} + \mathcal{G}))^{-n} P_0\|_\mathcal{H} \leq Ch$$

Convergence of order $O(h^p + h)$ \hspace{1cm} ($p =$ IE order)
Outline

Splitting methods for evolution equations

The differential Riccati equation

Implementation issues and results
Large-scale setting

Solving for large (dense) matrices:

\[ P(t) \in \mathbb{R}^{N \times N} \quad \text{with} \quad N \in [10^3, 10^7] \]

Storage?

Essential to use structural properties!

\textbf{Low-rank:} \quad P \approx zz^T \quad \text{with} \quad z \in \mathbb{R}^{N \times m}

Our splitting methods preserve low rank: \( \mathcal{M}_h z z^T = w w^T \)
Results: Mixed Lie

\[ \dot{P}(t) = A^T P(t) + P(t)A + C^T C - P(t)P(t) \]

\[ A = \nabla \cdot (\alpha \nabla u) - I, \quad \text{per. BC} \]

\[ \alpha(x) = 2 + 2 \cos 2\pi x \]

\[ C \sum_{k=0}^{\infty} a_k e^{2\pi ikx} = \sum_{k=0}^{4} a_k e^{2\pi ikx} \]

Discretization: 2001 points in space

Relative errors, \( \|M^n_h P_0 - P(nh)\|_{\text{Fro}} / \|P(nh)\|_{\text{Fro}} \)
Thank you

References on my webpage:
http://www.maths.lu.se/staff/tony-stillfjord/research/