Collection and overview of collision terms

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Abstract

In this document we present an overview of collision terms. These terms are used in kinetic modeling to describe the interactions between particles, i.e. the collisions. We limit ourselves to terms relevant to plasma physics. We list these terms in a unified way and make it easy to compare them. Additionally, we also look at the theoretical background of such collision operators and how they are designed. This is of interest, because our goal is to design numerical schemes for them.

Contents

Nomenclature

$\|\cdot\|$	(Euclidean) vector norm
α	distribution species (may be omitted in notation)
$\beta, \gamma, \ldots$	secondary particle species
ς	differential cross section
$\Delta\mathbf{v}$	velocity difference
$*$	excitation state
$\bullet$	ground state
λ	mean free path
$\mu, \nu, \ldots$	summation indices
ν	collision rate
Φ	observable
σ	cross section
$\varphi(\mathbf{x}, \mathbf{v})$	test function
ϑ	scattering angle
$\mathbf{F}$	(external) force
$\mathbf{u}$	relative velocity
$\mathbf{v}$	velocity coordinates
$\mathbf{x}$	space coordinates
ξ	speed normalized by the thermal speed
C	collision operator
e	elementary charge
f	phase space density
L	Landau number
m	single particle mass
n	particle density
P	probability of a particle changing properties during a collision
t	time
$v^{\pm}$	speed shifted by the threshold energy
W_{th}	threshold energy
v_T	thermal speed

1 General Background

At first, we want to look into the theoretical background that is necessary to study collision terms. These equations typically depend on time t, on three space dimensions $\mathbf{x}$ and on three velocity dimensions $\mathbf{v}$. Probability density functions called phase space densities $f : \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}} \to \mathbb{R}_{\geq 0}$ live in this world. On the following pages f directly conforms to the particle density. But in general f can be normalized by the total particle number.

In theory, the domain of f in velocity space is all of $\mathbb{R}^3$. But an unbounded domain is impractical for numerical simulations. Fortunately one can safely assume that f is 0 for large absolute velocities, i.e. speed. In the very least, the final cut-off velocity would be the speed of light. But in this manuscript, we don't go into the details for the relativistic case. Due to the existence of cut-off speeds, the velocity space can be assumed compact, which aids the discretization of this theoretically unbounded space. With this bounded velocity space boundary conditions become relevant for it.

There are examples, where an unbounded space domain is preferable, e.g. when modeling a star. But for the use cases we have in mind, i.e. simulating fusion relevant plasmas, we only look at a subset of space. Hence, we need to pay attention to the boundary conditions. When constructing these, we also have to mind how the velocity distribution looks like at the relevant boundary locations.

This model, at least when talking of the classical Vlasov equation, has some limits. So the particle density must be neither too small nor too high, i.e. there must be enough particles in a given volume so that one can still speak of matter. But too many particles are also not admitted, because there the presumption of binary collisions might become wrong or particles might even occupy the same space for periods of time. In the latter case using quantum mechanical methods would be necessary.

1.1 The Vlasov equation

The Vlasov equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v^\mu \frac{\partial f}{\partial x^\mu} + \frac{F^\mu}{m} \frac{\partial f}{\partial v^\mu} = 0 \tag{1}$$

can be used to model particle (of mass m) behavior under the external force $\mathbf{F}$. The downside of this model is that there is no short-range interaction between the particles. This changes when using the inhomogeneous Vlasov equation, i.e. the Boltzmann equation,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v^\mu \frac{\partial f}{\partial x^\mu} + \frac{F^\mu}{m} \frac{\partial f}{\partial v^\mu} = C(f, f_\beta)\,, \tag{2}$$

where the inhomogeneity is the so called collision term. It describes how particles of f (or f_α) interact with those of f_β and how f itself is consequently changed in the given phase space point. The collision operator depends on the particle distributions $f(t, \mathbf{x}, \mathbf{v})$ and $f_\beta(t, \mathbf{x}, \mathbf{v})$. Hence, it implicitly also has time, position and velocity as dependencies.

Note that the Einstein summation notation is used in these equations. It is utilized in this document in general. For cases, where additional indices are used next to those of a sum, the meaning of specific indices should be clear from the context.

The collision term C on the right-hand side of equation (2) is a composition of multiple distinct collision terms. Each models the interactions of particles represented by the given f with others, which are described in their own respective phase spaces. As such, the total collision term is the sum of all partial terms. Hence, $C(f, f_\beta, f_\gamma, \dots)$ depends on all these distributions.

In general, very few problems can be described only with the given Vlasov equation. Usually other restrictions lead to a system of (differential) equations. An example that is important for plasma physics are the *Vlasov–Maxwell* equations or, for the simpler non-relativistic case without any magnetic field the *Vlasov–Poisson* equations.

Principally there can be an additional term besides the collision operator C on the right-hand side. That would be a particle source/sink term, which is very similar to a collision term. And it does exactly what it is named for. In that sense, collision operators can also be viewed as particle sources and sinks in specific phase space points.

1.2 Collisions

First let us look into what a collision actually is. Within the black box of a collision we hide all the microscopic (quantum-)effects that happen when particles interact. All we allow ourselves to say is that the colliding particles had certain non-correlated properties before the collision. And after the interaction these properties, e.g. a specific velocity, have changed. In general, it is assumed that there are only binary collisions, i.e. all interactions are between two particles at a time and not more. These two ideas require the collisions to be very short in comparison to all other effects. Thus, we get limits to our model with respect to density, temperature etc. To get a better idea of such a collision, take a look at figure 1.

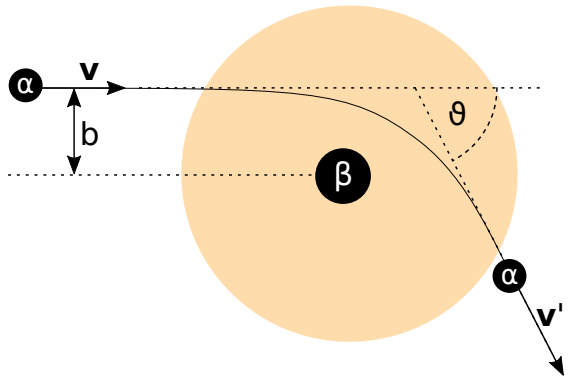

Figure 1: In this example a particle of species α is deflected by a particle of species β - a collision. Before colliding, the particle has the velocity $\mathbf{v}$ and afterwards $\mathbf{v}'$. In the plane of $\mathbf{v}$ and $\mathbf{v}'$ the particle α changes its direction by the angle of ϑ. b is the collision parameter, i.e. the distance between its trajectory in infinity and the parallel one directed to the targets center.

The very short range of a collision also implies that it is always localized at a point in space and that no collision can happen over a longer distance, i.e. between particles in different grid cells. Thus, one also gets a lower bound on the spatial grid size.

Consequently, in most cases, collision operators depend on the two particle distributions f_α and f_β. Collision terms change, if there are no particle sources or sinks, only the velocity distribution. If the particles are "collisionless", the particles' velocity distribution changes only due to influence of an external force. If the operator is inelastic it perturbs the thermal velocity distribution (i.e. a Maxwellian/Gaussian). Many other collision terms make the medium more thermalized.

Hence, we have several parameters, most of which are illustrated in figure 1, to describe binary collisions.

1. Under the assumption of binary collisions, we always define collision operators between two particle species. As such, we denote the species of the colliding particles with α and the targets correspondingly with β. If there is no specific index, one may assume within this document that α was simply omitted.

When combining multiple collision operators in a total one, obviously additional identifiers must be given. These are then chosen from the context.

2. Further, there is the collision parameter b. It is the distance of the colliding particle α's trajectory at infinity to the parallel trajectory directed to the scattering center. The collision parameter is shown in figure 1.

3. The particle changes its direction by the scattering angle ϑ.

4. The α particle has the velocity $\mathbf{v}$ before and $\mathbf{v}'$ after the collision. Hence, we define the velocity change as
$$\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}\,. \tag{3}$$

5. Further we define the relative velocity before the collision (i.e. in infinite distance) between two particles α and β as
$$\mathbf{u} = \mathbf{v}_\alpha - \mathbf{v}_\beta \tag{4a}$$
and the total relative speed as
$$u = \|\mathbf{u}\|\,. \tag{4b}$$

Note that we write the absolute value of a vector as
$$u = \|\mathbf{u}\|\,, \tag{5}$$
where
$$\|\cdot\|$$
is by default the *Euclidean norm*.

Any collision is an interaction between the collision partners. At least for elastic (i.e. the total energy is conserved) collisions, the particles move around the *center of mass* (COM). If the particle β is much heavier than α, then β is basically not affected. In that case, the COM is near β.

At least for all elastic interactions we assume that the probability is invariant to swapping incoming and outgoing velocities, i.e.
$$P((\mathbf{v}_\alpha, \mathbf{v}_\beta) \to (\mathbf{v}'_\alpha, \mathbf{v}'_\beta)) = P((\mathbf{v}'_\alpha, \mathbf{v}'_\beta) \to (\mathbf{v}_\alpha, \mathbf{v}_\beta))\,. \tag{6}$$
This is due to symmetries in trajectories. In some papers, for example Villani 2002, this is called *microreversible*.

A priori, no specific properties or limitations of collision operators can be assumed. They partially depend on the operators used to model the physics, which should obviously reflect reality. For example, it is quite common to linearize an operator if, e.g., a given property has only limited effect on the general system. But such techniques are used generally and follow the same rules as above.

1.3 Cross sections

This chapter is based on Landau, Lifschitz, and Ziesche 1997. Furthermore, the parameters and quantities defined in the list on p. 5 are used.

In physics differential cross sections describe the probability that a particle is deflected by a certain angle when colliding with another particle. The same proposition can not only be made

for a single particle, but also for a particle flux moving towards a collision target. Then one knows how much of that particle flux is spread and deflected into which directions.

Hence, the differential cross section represents the share of an input flux density j_0 that is scattered by the angle ϑ like

$$\varsigma(\vartheta, u) = \frac{j(u, \vartheta)}{j_0(u)}\,, \tag{7}$$

where u is the relative speed. In a general case one also has to observe the azimuthal angle and not only the polar angle ϑ. However, the azimuthal angle can be neglected in this case due to symmetries of the acting forces.

1.3.1 Central potential

For a central potential the scattering angle ϑ only depends on the collision parameter b and the relative speed u. In the following deliberations u is a parameter. As already mentioned, these scattering processes are axial symmetric, i.e. they are independent of the azimuthal angle φ.

The following deduction of the differential cross section is aided by figure 2.

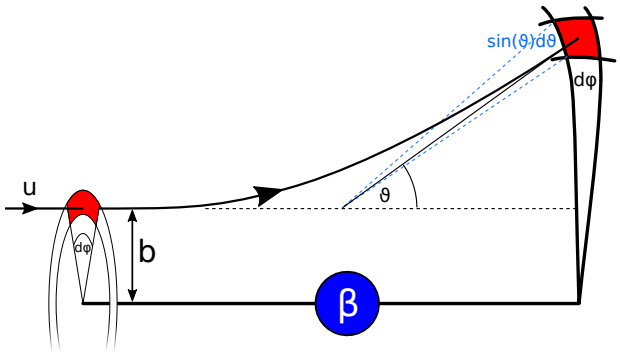

Figure 2: If particles (red) move to the collision target (β) they are scattered. Under a central force the scattering angle only depends on the collision parameter b and the relative speed u. This can be used to model the corresponding differential cross section.

We assume that there is a homogeneous influx of particles. Then the share of particles deflected by the angle ϑ is proportional to the area that corresponds to a given b. That area element is perpendicular to the inflow direction and is given as

$$\mathrm{d}\sigma = b\,\mathrm{d}b\,\mathrm{d}\varphi\,. \tag{8}$$

Then we interpret the collision parameter

$$b(\vartheta) \tag{9}$$

as a function of the corresponding scattering angle ϑ. From equations (8) and (9) we obtain

$$\mathrm{d}\sigma = b(\vartheta)\left|\frac{\partial b}{\partial \vartheta}(\vartheta)\right|\mathrm{d}\vartheta\,\mathrm{d}\varphi\,. \tag{10}$$

Note that we use the absolute value of the derivative. This is due to the fact that (depending on an attractive or repulsive central force) the term can be negative, but it has to be positive in equation (8).

Consequently, the solid angle element corresponding to the outgoing particles is given by

$$\mathrm{d}\Omega = \sin(\vartheta)\mathrm{d}\vartheta\mathrm{d}\varphi\,. \tag{11}$$

Thus, we rewrite equation (10) as

$$\mathrm{d}\sigma = \frac{b(\vartheta)}{\sin(\vartheta)} \left| \frac{\partial b}{\partial \vartheta}(\vartheta) \right| \mathrm{d}\Omega \tag{12}$$

and define the *differential cross section* as

$$\varsigma(\vartheta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b(\vartheta)}{\sin(\vartheta)} \left| \frac{\partial b}{\partial \vartheta}(\vartheta) \right| . \tag{13}$$

We usually use a predefined function for $\varsigma(\vartheta)$ that describes the scattering rate of the colliding particles at a certain angle. A well-known differential cross section is provided by the *Rutherford scattering*, which is given in equation (74). Other (less fundamental) cross sections are obtained through experiments or via a theoretical approach. They can be found, for example, in collections like Zerkin 2017. There also exist some publications that collect cross sections and describe the corresponding collisions in a bigger setting. Examples for such collections can be found in Beyer and V.P. Shevelko 2002; Janev, Langer, et al. 1987; Janev, Presnyakov, and V. Shevelko 1985; Janev, Reiter, and Samm 2004.

1.3.2 The total cross section

When integrating over the sphere

$$\sigma = \int_0^{2\pi} \int_0^{\pi} \varsigma(\vartheta) \sin(\vartheta) \mathrm{d}\vartheta \mathrm{d}\varphi \tag{14}$$

one gets the total cross section. It essentially is an area perpendicular to the direction of motion. If two particles lie within it, then a collision occurs. Hence, it is often compared to solid balls that hit each other. Then the total cross section is just the area of the circle of which the radius is the sum of the individual balls' radii. This is depicted in figure 3.

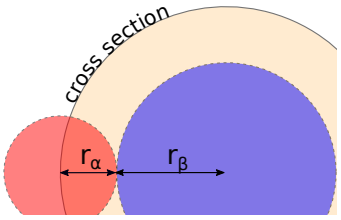

Figure 3: The cross section for hard balls is just the area of the circle that can be spanned when they touch.

In physics there are some simple but well-known estimates connecting the total cross section σ to other quantities. With the particle density n and the mean free path λ, i.e. the mean distance a particle travels between collisions, the total cross sections is

$$\sigma = \frac{1}{n\lambda} . \tag{15}$$

With an additional relative speed u the collision rate is

$$\nu = n\sigma u = \frac{u}{\lambda} . \tag{16}$$

1.4 Deriving physical properties

All physical properties that can be derived from the phase space density are strongly connected to the corresponding (velocity) moments. For the following examples we only look at the velocity component of the distribution. As f is a probability distribution, the derived properties are quite often moments. Thus, the 0^{th} moment

$$n(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v} \tag{17}$$

is the particle density $n(t, \mathbf{x})$. In the same sense the first velocity moment

$$(n\mathbf{u})(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v} \tag{18}$$

is the momentum density, where $\mathbf{u}(t, \mathbf{x})$ is the mean velocity. Similarly, one can compute electrical and mass density distributions as

$$\rho_q(t, \mathbf{x}) = qn(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} q f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v} \tag{19}$$

and

$$\rho_m(t, \mathbf{x}) = mn(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} m f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v}\,, \tag{20}$$

where q and m are the individual particles corresponding charge and mass. Analog to equation (18) the first moments follow as well, i.e. the electrical current density

$$\mathbf{j}(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} q\mathbf{v} f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v} \tag{21}$$

and the momentum density as

$$(mn\mathbf{u})(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} m\mathbf{v} f(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v}\,. \tag{22}$$

Next we define the second moments, the velocity covariance matrix, which coincides with the pressure tensor as

$$p_{ij}(t, \mathbf{x}) = m \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v})(v^i - u^i)(v^j - u^j) \ \mathrm{d}\mathbf{v}. \ ^{1} \tag{23}$$

Hence, half of the trace of that matrix corresponds to the temperature T, which can be defined directly via the variance as

$$(nT)(t, \mathbf{x}) = (n\frac{1}{2}mv_T^2)(t, \mathbf{x}) = \frac{m}{2} \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) \|\mathbf{v} - \mathbf{u}\|^2 \ \mathrm{d}\mathbf{v}\,, \tag{24}$$

[1] $\ldots \widehat{=} m\Delta v^i \Delta v^j \ n = m\frac{\Delta v^i \Delta v^j}{V} = m\frac{\Delta v^i \Delta v^j}{\Delta x^i A^i_\perp} = m\frac{\cancel{\Delta x^i}}{\Delta t}\frac{\Delta v^j}{\cancel{\Delta x^i} A^i_\perp} = \frac{F^j}{A^i_\perp}$

where v_T is the thermal speed. Note that we do not write the Boltzmann constant k. In general, the temperature is the intrinsic kinetic energy of a (moving) medium. Therefore, the thermodynamic temperature is defined as the variance of the particles' Gaussian velocity distribution.

Among others, there are further plasma properties like the heat flux

$$\mathbf{Q}(t,\mathbf{x}) = \frac{m}{2}\int\limits_{\mathbb{R}^3_\mathbf{v}} f(t,\mathbf{x},\mathbf{v})\|\mathbf{v}-\mathbf{u}\|^2\mathbf{v}\ \mathrm{d}\mathbf{v}\,. \tag{25}$$

1.4.1 The general idea

In general, one chooses the test function $\varphi(\mathbf{x},\mathbf{v})$ to make the corresponding measurement as

$$\phi(t,\mathbf{x}) = \int\limits_{\mathbb{R}^3_\mathbf{v}} \varphi(\mathbf{x},\mathbf{v}) f(t,\mathbf{x},\mathbf{v})\ \mathrm{d}\mathbf{v}\,, \tag{26}$$

where possible choices for φ can be read from equations (17) to (25).

When integrating this $\phi(t,\mathbf{x})$ over the entirety of the space domain, one gets the macroscopic observable

$$\Phi(t) = \int\limits_{\mathbb{R}^3_\mathbf{x}} \phi(t,\mathbf{x})\ \mathrm{d}\mathbf{x}\,. \tag{27}$$

So, for example with $\varphi = 1$, one gets the particle density $n = \phi(t,\mathbf{x})$ and the total particle number $N = \Phi(t)$.

It is worth noting that Villani 2002 defines $\varphi(\mathbf{v})$ as an *arbitrary continuous function of the velocity* $\mathbf{v}$. Even though that might be true for most use cases, we see no reason to restrict φ in that way. Hence, we defined it here a bit more generally.

1.4.2 Multiple species

But until now, all these observations describe only the single species case. To obtain overall measurements with the set S of species, it is necessary to combine them like

$$\phi(\mathbf{x},\mathbf{v}) = \sum_{s\in S}\phi_s(\mathbf{x},\mathbf{v}) = \int\limits_{\mathbb{R}^3_\mathbf{v}} \sum_{s\in S}\varphi_s(\mathbf{x},\mathbf{v}) f_s(t,\mathbf{x},\mathbf{v})\ \mathrm{d}\mathbf{v}\,. \tag{28}$$

For example, if there are electrons and Z-times positively charged ions the current density can be obtained with

$$\mathbf{j}(t,\mathbf{x}) = \int\limits_{\mathbb{R}^3_\mathbf{v}} \mathbf{v}e(Z\cdot f_i(t,\mathbf{x},\mathbf{v}) - f_e(t,\mathbf{x},\mathbf{v}))\ \mathrm{d}\mathbf{v}\,. \tag{29}$$

There e is the elementary charge.

1.5 The conservation of properties

When designing numerical schemes, it is always interesting to know the conservation of properties. Naturally, this topic has already been looked into. We base this section on the existing overviews in Sonnendrücker 2015, p. 63 and Villani 2002, p. 30.

So, what does it mean to conserve properties? All in all, that certain physical observables do not change with time, they are invariant. Typically, these are the properties described in

section 1.4, one speaks of the probability moments in velocity space over the entire domain, i.e. conservation of the particle number, the various fluxes and energy. The most important ones are described in section 1.4.

So we have to investigate whether these (macroscopic) properties (cf. with equation (27)) are preserved with time. And if they are not, it is helpful to know how they behave. We basically want to know the properties of

$$\dot{\Phi}(t)\,. \tag{30}$$

We defined $\varphi(\mathbf{x},\mathbf{v})$ to be independent of t. Thus, we use the inhomogenous Vlasov equation (2) to compute equation (30) as

$$\begin{aligned}\dot{\Phi}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3_\mathbf{x}} \phi(t,\mathbf{x})\ \mathrm{d}\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3_\mathbf{x}}\int\limits_{\mathbb{R}^3_\mathbf{v}} \varphi(\mathbf{x},\mathbf{v}) f(t,\mathbf{x},\mathbf{v})\ \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\ &= \int\limits_{\mathbb{R}^3_\mathbf{x}}\int\limits_{\mathbb{R}^3_\mathbf{v}} \varphi(\mathbf{x},\mathbf{v})\left(\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathbf{v}}\right)\ \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} = \int\limits_{\mathbb{R}^3_\mathbf{x}}\int\limits_{\mathbb{R}^3_\mathbf{v}} \varphi(\mathbf{x},\mathbf{v}) C(f)\ \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{31}$$

So, the properties are conserved, if these integrals are zero. That principally also depends on the boundary conditions, which can therefore prevent the conservation of properties if, e.g., they act reflective or are a source. But let's assume for the following deliberations that there are no boundary conditions or they are nice, i.e. we have an infinite domain or periodic boundary conditions.

Hence, for an observable with the test function φ to be conserved, the integrals in equation (31) must be zero. For the term on the right-hand side that highly depends on the collision term C. But naturally also the test function itself is important: if it does not depend on $\mathbf{v}$, that integral would be zero, because practically all collision terms are localized in space and only change the densities in velocity space. But practically all the interesting observables in section 1.4 depend on $\mathbf{v}$.

A similar statement can be made for the part left of it. So, a steady advection in the space domain does not change the velocity moments. Therefore, that integral is zero for such test functions if there is no external force $\mathbf{F}$.

The homogeneous Vlasov equation without an external force conserves the mentioned velocity moments. But only with appropriate boundary conditions.

We also remark that some collision terms are designed not to conserve, for example, the total energy. The majority of these are inelastic/dissipative collision operators, which take by definition some of the particles' energy, i.e. the second moment. For such collision operators $\dot{\Phi}$ and also $\dot{\phi}$ should reflect the energy dissipating from the modeled system. Another straightforward example of non-conserving operators are particle sources and sinks.

2 Modeling collision operators

In the following, we present basic models that are often used to create collision terms for the simulation of plasmas.

Collision terms are on the right-hand side of equation (2) and therefore also describe a rate. In some publications (like Karney 1986, p. 187) a vector (flux) field $\mathbf{S}(t,\mathbf{x},\mathbf{v})$ is used to describe collisions, such that

$$C(t,\mathbf{x},\mathbf{v}) = -\frac{\partial}{\partial v^\mu}\mathbf{S}^\mu\,. \tag{32}$$

This is advantageous in some cases, especially when transforming the coordinate system. Karney 1986 says that this can be done due to the prevalent small angle scattering. This condition is sufficient but not necessary. Thus, all operators that do not scatter to a preferred direction are viable.

2.1 BGK-type collision operators

The Bhatnagar–Gross–Krook (BGK) type operators, which are explained in Bhatnagar, Gross, and Krook 1954, p. 514, are pretty simple. Basically, there is a known equilibrium from which the particles deviate. A (linear) BGK-type operator enforces the equilibrium represented by the phase space distribution f_{eq}

$$C(t, \mathbf{x}, \mathbf{v}) = \nu(v) \left(f_{eq}(t, \mathbf{x}, \mathbf{v}) - f(t, \mathbf{x}, \mathbf{v}) \right), \tag{33}$$

with speed $v = \|\mathbf{v}\|$ and collision rate $\nu(v)$.

But this expression may be problematic, because physical properties are not necessarily conserved. Let's look at the particle density as defined in equation (17). With a simple constant collision rate $\nu(v) = \nu_{const} = const$ one gets

$$\dot{n}(t, \mathbf{x}) = \int\limits_{\mathbb{R}^3_{\mathbf{v}}} C(t, \mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v} = \nu_{const} \int\limits_{\mathbb{R}^3_{\mathbf{v}}} \left(f_{eq}(t, \mathbf{x}, \mathbf{v}) - f(t, \mathbf{x}, \mathbf{v}) \right) \ \mathrm{d}\mathbf{v} = \nu_{const} \left(n_{eq}(t, \mathbf{x}) - n(t, \mathbf{x}) \right). \tag{34}$$

For particle number conservation one needs $\dot{n}(t, \mathbf{x}) = 0$. Hence, one can fix it by assuming a perturbation in velocity space only. This leads to the modified collision operator

$$C(t, \mathbf{x}, \mathbf{v}) = \nu_{const} \left(\frac{n(t, \mathbf{x})}{n_{eq}(t, \mathbf{x})} f_{eq}(t, \mathbf{x}, \mathbf{v}) - f(t, \mathbf{x}, \mathbf{v}) \right). \tag{35}$$

In most cases it is necessary to have a relaxation time τ, which is independent of the velocity. According to the source, one can choose a suitable average collision time. A simple ansatz would be to use the collision rate as presented in equation (16).

2.2 The Boltzmann collision operator

A collision operator can be viewed as the difference of all the particles that leave and enter a specific phase space point in a given amount of time. Hence, we can use simple rates to describe collisions. This rate equation looks like

$$\begin{aligned} C_{\alpha\beta}(t, \mathbf{x}_\alpha, \mathbf{v}_\alpha) = &\iint\limits_{\mathbb{R}^3_{\mathbf{v}} \mathbb{R}^3_{\mathbf{v}}} P(\mathbf{v}'_\alpha, \mathbf{v}_\alpha, \mathbf{v}'_\beta) f_\alpha(t, \mathbf{x}_\alpha, \mathbf{v}'_\alpha) f_\beta(t, \mathbf{x}_\alpha, \mathbf{v}'_\beta) \mathrm{d}\mathbf{v}'_\alpha \mathrm{d}\mathbf{v}'_\beta \\ &- \iint\limits_{\mathbb{R}^3_{\mathbf{v}} \mathbb{R}^3_{\mathbf{v}}} P(\mathbf{v}_\alpha, \mathbf{v}'_\alpha, \mathbf{v}'_\beta) f_\alpha(t, \mathbf{x}_\alpha, \mathbf{v}_\alpha) f_\beta(t, \mathbf{x}_\alpha, \mathbf{v}'_\beta) \mathrm{d}\mathbf{v}'_\alpha \mathrm{d}\mathbf{v}'_\beta \,, \end{aligned} \tag{36}$$

where $P(\mathbf{v}'_\alpha, \mathbf{v}_\alpha, \mathbf{v}'_\beta)$ is the probability of a particle of type α to change its velocity from $\mathbf{v}'_\alpha$ to $\mathbf{v}_\alpha$ when hitting a particle of type β with velocity $\mathbf{v}'_\beta$. Actually it would be $\widetilde{P}(\mathbf{v}'_\alpha, \mathbf{v}_\alpha, \mathbf{v}'_\beta, \mathbf{v}_\beta)$, i.e. the probability for the velocities to change like

$$(\mathbf{v}'_\alpha, \mathbf{v}'_\beta) \to (\mathbf{v}_\alpha, \mathbf{v}_\beta) \,. \tag{37}$$

But as the α collision term does not really care what happens to the β particles, one can simply ignore it and set that partial probability to 1. Also, note that equation (36) has been written in a generalized manner. That integral

$$C_{\alpha\beta}(t,\mathbf{x}_\alpha,\mathbf{v}_\alpha) = \iint\limits_{\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}} \big(P(\mathbf{v}'_\alpha,\mathbf{v}_\alpha,\mathbf{v}'_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}'_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}'_\beta) - P(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}'_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}'_\beta)\big)\,\mathrm{d}\mathbf{v}'_\alpha\mathrm{d}\mathbf{v}'_\beta \tag{38}$$

is the same as equation (36), but the $\mathbf{v}_\beta$ integral has been adjusted.

Keep in mind that these collision terms do not change f_β, for which an equivalent term $C_{\beta\alpha}$ would be necessary in another Vlasov equation for f_β. This probability P can also be seen as a positive definite kernel, which is then called the *Boltzmann collision kernel* (cf. with Villani 2002).

2.2.1 Conservation of properties

Now we use the basic concepts presented in section 1.5 and analyze for which test functions $\varphi(\mathbf{x},\mathbf{v})$ a Boltzmann type collision operator is conserving the observable for a generic phase space density function $f(t,\mathbf{x},\mathbf{v})$.

The major steps of the following calculations can be found in Villani 2002, p. 30, but in more detail in Cercignani, Illner, and Pulvirenti 2013, p. 33. Also, take note that this ansatz will only work for elastic processes. But that will become clear within the next few lines anyhow.

So we start by entering equation (36) into equation (31) to obtain

$$\dot\phi(t,\mathbf{x}) = \int\limits_{\mathbb{R}^3_\mathbf{v}} \varphi(\mathbf{x},\mathbf{v}_\alpha) \left(\iint\limits_{\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}} P(\mathbf{v}'_\alpha,\mathbf{v}_\alpha,\mathbf{v}'_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}'_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}'_\beta)\;\mathrm{d}\mathbf{v}'_\alpha\mathrm{d}\mathbf{v}'_\beta - \iint\limits_{\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}} P(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}_\beta)\;\mathrm{d}\mathbf{v}'_\alpha\mathrm{d}\mathbf{v}_\beta \right) \mathrm{d}\mathbf{v}_\alpha\,. \tag{39}$$

Hence, we interchange the variables in the first part of that integral like $(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}_\beta,\mathbf{v}'_\beta) \to (\mathbf{v}'_\alpha,\mathbf{v}_\alpha,\mathbf{v}'_\beta,\mathbf{v}_\beta)$ and obtain

$$\dot\phi(t,\mathbf{x}) = \iiint\limits_{\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}} P(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}_\beta)(\varphi(\mathbf{x},\mathbf{v}'_\alpha) - \varphi(\mathbf{x},\mathbf{v}_\alpha))\;\mathrm{d}\mathbf{v}'_\alpha\mathrm{d}\mathbf{v}_\beta\;\mathrm{d}\mathbf{v}_\alpha\,. \tag{40}$$

The next part is valid, if the resulting $\dot\phi$ does not change, when swapping $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$, and thus also $\mathbf{v}'_\alpha$ and $\mathbf{v}'_\beta$. Then we can also assume that

$$P(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}_\beta,\mathbf{v}'_\beta) = P(\mathbf{v}_\beta,\mathbf{v}'_\beta,\mathbf{v}_\alpha,\mathbf{v}'_\alpha)\,. \tag{41}$$

There, we denoted the probability depending on the unnecessary full set of variables to make it easier to understand what actually happens.

Thus, we can write the same integral as

$$\dot\phi(t,\mathbf{x}) = \frac{1}{2} \iiint\limits_{\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}\,\mathbb{R}^3_\mathbf{v}} P(\mathbf{v}_\alpha,\mathbf{v}'_\alpha,\mathbf{v}_\beta) f_\alpha(t,\mathbf{x}_\alpha,\mathbf{v}'_\alpha) f_\beta(t,\mathbf{x}_\alpha,\mathbf{v}'_\beta) \cdot \big(\varphi(\mathbf{x},\mathbf{v}'_\alpha) + \varphi(\mathbf{x},\mathbf{v}'_\beta) - \varphi(\mathbf{x},\mathbf{v}_\alpha) - \varphi(\mathbf{x},\mathbf{v}_\beta)\big)\;\mathrm{d}\mathbf{v}'_\alpha\mathrm{d}\mathbf{v}_\beta\;\mathrm{d}\mathbf{v}_\alpha\,. \tag{42}$$

Hence, we can deduce conservation of the property given by $\varphi(\mathbf{x}, \mathbf{v})$ if for all $\mathbf{x}$

$$\varphi(\mathbf{x}, \mathbf{v}'_\alpha) + \varphi(\mathbf{x}, \mathbf{v}'_\beta) = \varphi(\mathbf{x}, \mathbf{v}_\alpha) + \varphi(\mathbf{x}, \mathbf{v}_\beta)\,. \tag{43}$$

2.3 The Fokker–Planck equation

The Fokker–Planck equation is used often in the form

$$C_\alpha = \frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = -\frac{\partial}{\partial v^\mu}\left(f_\alpha \cdot \langle \Delta v^\mu \rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial v^\mu \partial v^\nu}\left(f_\alpha \cdot \langle \Delta v^\mu \Delta v^\nu \rangle\right). \tag{44}$$

Whereas the first part of equation (44) is simply a divergence form, the second part is a little bit more complicated. In the literature, e.g. Karney 1986, they use the notation

$$\nabla\nabla = \frac{\partial^2}{\partial v^\mu \partial v^\nu}\,. \tag{45}$$

We first explain what this operator actually does. It reduces the second order tensor to one of order zero, i.e. a scalar value. It is similar to the divergence, as the divergence reduces a tensor's order by one. Hence, we have here a "double" divergence on the tensor field $\chi(\mathbf{v})$ that can actually be written as

$$\frac{\partial^2}{\partial v^\mu \partial v^\nu}\chi^{\mu\nu} = \nabla \cdot \left(\nabla \cdot \chi^T\right). \tag{46}$$

In our case of a symmetric matrix, $\chi^T = \chi$, this becomes even simpler.

In equation (44) Δv is the change of velocity as defined in equation (3). The collision of every particle of α also depends on f_β, the target species distribution. Hence,

$$\langle \boldsymbol{\Delta}\mathbf{v} \rangle_\alpha(t, \mathbf{x}, \mathbf{v}_\alpha) \tag{47}$$

is the mean velocity change for a particle α. That mean is obtained by averaging over the distribution f_β.

In the often much more convenient spherical coordinate system ($\Omega \equiv (\phi, \vartheta), \mathrm{d}\Omega = \sin(\vartheta)\mathrm{d}\phi\mathrm{d}\vartheta$) the same operation can be done in the following way. With the relative speed $u(\mathbf{v}_\beta, \mathbf{v}_\alpha) = \|\mathbf{v}_\beta - \mathbf{v}_\alpha\|$ between the particles and a given differential cross-section $\varsigma(u, \vartheta)$ (see section 1.3) one computes the averages like

$$\langle \Delta v^\mu \rangle_\alpha(t, \mathbf{x}, \mathbf{v}_\alpha) = \sum_\beta \int_{\mathbb{R}^3} f_\beta(t, \mathbf{x}, \mathbf{v}_\beta) u \int_{S^2} \varsigma(u, \vartheta)\ \Delta v^\mu_\alpha(u, \vartheta)\ \mathrm{d}\Omega\ \mathrm{d}\mathbf{v}_\beta \tag{48a}$$

and

$$\langle \Delta v^\mu \Delta v^\nu \rangle_\alpha(t, \mathbf{x}, \mathbf{v}_\alpha) = \sum_\beta \int_{\mathbb{R}^3} f_\beta(t, \mathbf{x}, \mathbf{v}_\beta) u \int_{S^2} \varsigma(u, \vartheta)\ \Delta v^\mu_\alpha(u, \vartheta)\Delta v^\nu_\alpha(u, \vartheta)\ \mathrm{d}\Omega\ \mathrm{d}\mathbf{v}_\beta\,. \tag{48b}$$

The additional u is used equivalently to equation (16). Note that both the differential cross section as well as the velocity difference $\Delta\mathbf{v}$ have to be known a priori. In the example detailed in section 4, the former one is defined in equation (74) and the latter one in equation (82).

2.3.1 Conservation of properties

Once again we investigate the conditions the Fokker–Planck type collision operator and an observable must adhere to such that the corresponding property is conserved. Hence, we multiply equation (44) once again with the test function $\varphi(\mathbf{x}, \mathbf{v})$ (cf. with section 1.4) and integrate. We get

$$\dot{\Phi} = \int\limits_{\mathbb{R}^3_{\mathbf{x}}} \int\limits_{\mathbb{R}^3_{\mathbf{v}}} \left(-\frac{\partial}{\partial v^\mu} \left(f_\alpha \cdot \langle \Delta v^\mu \rangle \right) + \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial v^\nu} \left(f_\alpha \cdot \langle \Delta v^\mu \Delta v^\nu \rangle \right) \right) \varphi(\mathbf{x}, \mathbf{v}) \ \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \,. \tag{49}$$

Integrating by parts delivers

$$\dot{\Phi} = \int\limits_{\mathbb{R}^3_{\mathbf{x}}} \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f_\alpha \left(\langle \Delta v^\mu \rangle \frac{\partial \varphi}{\partial v^\mu} + \frac{1}{2} \langle \Delta v^\mu \Delta v^\nu \rangle \frac{\partial^2 \varphi}{\partial v^\mu \partial v^\nu} \right) \ \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \,. \tag{50}$$

Here the test function must be twice differentiable in $\mathbf{v}$. In some cases – if the test function φ has no compact support – it is necessary for f to tend to 0 for $\mathbf{x}, \mathbf{v} \to \infty$.

From this we can already deduce that all Fokker–Planck type collision operators are particle number conserving, because the derivatives of the corresponding test function $\varphi(\mathbf{x}, \mathbf{v}) = 1$ are zero. More propositions can only be taken with the knowledge of the specific $\langle \Delta v^\mu \rangle$ and $\langle \Delta v^\mu \Delta v^\nu \rangle$.

2.4 Approximations

Under the right circumstances it is possible to assume certain simplifications for the target particle distribution. Such assumptions may also be valid even if identical particles (from the same phase space distribution) collide with each other. Karney 1986, p. 194 suggests the following simplifications.

2.4.1 Isotropic background distribution

If the background medium's velocity distribution $f_\beta(t, \mathbf{x}, \mathbf{v}) = \tilde{f}_\beta(t, \mathbf{x}, v)$ is isotropic, many of the above equations become simpler. Practically all relevant collision operators integrate over the entire velocity domain. Hence, a reduction of dimension is especially advantageous, also to have less global operations. This can also be observed for *Fokker–Planck* type equations: there are not only fewer integrals as can be easily seen in, e.g., equation (48a), but equation (44) itself may become much easier.

Unsurprisingly, one can only profit of many of the advantages that come with an isotropic background distribution in a fitting coordinate system. In this case this obviously means spherical coordinates.

2.4.2 High-velocity limit

Another useful approximation can be applied if the velocity distribution of one particle species is much higher than the others. An example for such a setting is particle beam plasma heating. There, a high velocity particle beam is inserted into the plasma and the beams kinetic energy is transmitted through collisions.

But such settings do not automatically lead to simpler equations. That also depends on other circumstances: the collision probabilities. So, basically the probability $P(\mathbf{v}_\alpha, \mathbf{v}_\beta)$ of particles of α and β colliding with their respective velocities. In many cases one uses a fitting differential cross section (see section 1.3) for such a probability. Hence, possible simplifications also depend on its structure and the used coordinate system.

2.4.3 Maxwellian background

A thermalized medium is called Maxwellian and has therefore a Gaussian velocity distribution. In many cases, if one particle species is much denser or has many more intrinsic collisions, one can safely assume that it is thermalized.

2.4.4 Linearization

Often the phase space density is expanded into a series of one kind or another. It is not uncommon to truncate the series after the linear term. In such a case also the collision operators can be expanded accordingly. A popular choice is to use *Legendre polynomials* as in Ginzburg and Gurevich 1960. This approach is exemplified below in section 3.1.

At this point it is also worthy to mention the *Chapman–Enskog–Braginskii expansion*, which is introduced in Braginskii 1965, p. 236 and Chapman, Cowling, and Burnett 1990, p. 134. There the exact solution is approximated with the expansion

$$f = f_0 + f_1 + f_2 + \dots . \tag{51}$$

The higher order terms have to be small in comparison to the lower ones. But that is true for every perturbation ansatz.

3 Inelastic collision terms

Inelastic events are those, in which energy is taken from the system. Even though inelasticity includes both effects where the particles gain and lose energy, we will look for now at the latter.

So if we talk about energy dissipating effects, the source and target particles change. The energy necessary to change is usually taken from the movement, i.e. the overall kinetic energy is reduced. Here we mostly talk of electrons colliding with neutral atoms. Examples for such inelastic processes are

ionization an electron is freed from its shell

electronic excitation an electron is elevated into a higher shell

vibrational excitation vibration states are stronger for molecules

rotational excitation rotational excitation is only relevant for molecules.

For some inelastic processes, e.g. ionization, the particles change or new ones appear. Thus, the corresponding collision terms act as particle sources and sinks, i.e. the particle numbers have to change. In other cases like the electronic excitation the particle numbers are invariant. In that case, one often assumes that the particle does not change its extrinsic properties and that the excitation state has a short half-life, i.e. the particle/atom emits a photon and returns to the ground state.

But it is common to all these excitation processes that the necessary energy is specific for each type. Hence, below this threshold of energy (e.g. the colliding electron is too slow), nothing happens. So let's define this specific kinetic energy threshold with the corresponding speed as

$$W_{th} = \frac{1}{2} m_\alpha v_{th}^2 \,. \tag{52}$$

In parts of the literature this threshold energy is defined as

$$W_{th} = \hbar\omega \,, \tag{53}$$

which is the energy of the photon with frequency ω that corresponds to that specific electronic excitation.

Note equation (5), where the speed was defined as the norm of the velocity as

$$v = \|\mathbf{v}\|\,, \tag{54}$$

which is consequently used here as well.

We continue with defining the shifted speeds as

$$v^{\pm} = \sqrt{v^2 \pm v_{th}^2}\,. \tag{55}$$

If a particle has the speed of v, it would have had v^+ before its last inelastic collision and v^- after another one. This notation is also applicable for vectors, where all velocity that coincide with that speed span a subspace. Consequently, if there is a function that depends on a velocity, but the variable is written as a speed, this indicates

$$f(t, \mathbf{x}, v) = \int\limits_{S^2} f(t, \mathbf{x}, \mathbf{v})\ \mathrm{d}\Omega\,. \tag{56}$$

Obviously this changes with the actual dimension of the velocity space. Here, Ω represents as before the solid angle.

In the following models we use $\bullet$ and $*$ to denote ground and excitation states respectively. If there is no special mark, the ground state can be safely assumed.

3.1 Ginzburg's model

In Ginzburg and Gurevich 1960, p. 128 a collision term for inelastic electronic collisions with neutral particles is described. They expand the phase space density function $f(t, \mathbf{x}, \mathbf{v})$ in $\mathbf{v}$ with zero order spherical harmonics, i.e.

$$f(t, \mathbf{x}, \mathbf{v}) = \sum_{k=0}^{\infty} P_k(\cos(\vartheta))\hat{f}_k(t, \mathbf{x}, v)\,. \tag{57}$$

There, P_k are the Legendre polynomials, ϑ is an angle to a preferred direction and $\hat{f}$ is a dimension reduced version of the phase space density f. Hence, such an expansion is only possible if you can define a direction to refer to. In plasmas one can often use the electric or magnetic field lines to align to. In this expansion the third velocity dimension can be neglected due to symmetry. If necessary, it is not too complicated to extend the model to depend on, e.g., an angle φ.

In this context it is common to define

$$\mu = \cos(\vartheta) = \frac{v_{\parallel}}{v}\,, \tag{58}$$

which consequently can also be described as the component of the velocity parallel to the preferred axis, normalized by the speed. Usually this ansatz is used, if the terms of higher order are small in comparison. Thus, most often one only considers the terms of order 0 and 1, and consequently obtains

$$f(t, \mathbf{x}, v, \mu) = f_0(t, \mathbf{x}, v) + \mu f_1(t, \mathbf{x}, v) \tag{59a}$$

and also

$$C(t, \mathbf{x}, v, \mu) = C_0(t, \mathbf{x}, v) + \mu C_1(t, \mathbf{x}, v)\,. \tag{59b}$$

In that sense, the isotropic f_0 can also be seen as the mean

$$f_0(t, \mathbf{x}, v) = \frac{1}{2} \int_{-1}^{1} f(t, \mathbf{x}, v, \mu) \mathrm{d}\mu \,. \tag{60}$$

As already mentioned, Ginzburg and Gurevich 1960 model electrons (also based on this expansion) in an electric field, for which interesting collision operators are presented. Their approximations lead to different models for different temperature regimes:

For high kinetic energies

$$\frac{1}{2} m v^2 \gg W_{th}$$

they obtain the following collision term

$$C_0(t, \mathbf{x}, v) = -\frac{1}{2v^2} \frac{\partial}{\partial v} \left(v^2 r_{in}(v) \left(\frac{T_{in}}{m_e} \frac{\partial f_0}{\partial v} + v f_0 \right) \right) \tag{61}$$

with the energy loss per unit time

$$r_{in}(v) = \frac{2 W_{th}}{m_e v^2} (n_{in}^{\bullet} - n_{in}^{*}) v \int \varsigma_{in} \, \mathrm{d}\Omega \tag{62}$$

and the effective temperature

$$T_{in} = \frac{W_{th}}{2} \frac{n_{in}^{\bullet} + n_{in}^{*}}{n_{in}^{\bullet} - n_{in}^{*}} \tag{63}$$

with the particle densities for the ground state ($n_{in}^{\bullet}$) and the excited state (n_{in}^{*}). The first order term C_1 comes along as

$$C_1(t, \mathbf{x}, v) = \nu_{in}(v) f_1 = (n_{in}^{\bullet} + n_{in}^{*}) \cdot v \int (1 - \mu) \; \varsigma_{in} \cdot f_1 \; \mathrm{d}\Omega \,. \tag{64}$$

For energies slightly above the threshold

$$\frac{1}{2} m v^2 = W_{th} + \epsilon \tag{65}$$

Ginzburg and Gurevich 1960 assume $n_{in}^{\bullet} \gg n_{in}^{*}$ and describe the zero-order term of the collision operator as

$$C_0(t, \mathbf{x}, v) = \nu_{in}(v) f_0 = n_{in}(\mathbf{x}) v \sigma(v) f_0 \tag{66}$$

and the consequent first-order term as

$$C_1(t, \mathbf{x}, v) = \nu_{in}(v) f_1 \,. \tag{67}$$

For the definition of ν, see equation (16).

3.2 Morse–Riemann model

Riemann 1992, p. 46 modifies the inelastic collision term introduced in Morse, Allis, and Lamar 1935 slightly and obtains with equation (55)

$$\begin{aligned} C_0(t,\mathbf{x},\mathbf{v}) = n^{\bullet}_{in}(t,x)\left(\frac{{v^+}^2}{v}\sigma^{\bullet}(v^+)\tilde{f}^{\bullet}(t,v^+) - v\sigma^{\bullet}(v)\tilde{f}^{\bullet}(t,v)\right) \\ + \frac{g^{\bullet}}{g^*}n^*_{in}\left(v\sigma^*\tilde{f}^{\bullet}(t,v^-) - \frac{{v^+}^2}{v}\sigma^*(v^+)\tilde{f}^{\bullet}(t,v)\right). \end{aligned} \tag{68}$$

As before, $\bullet$ and $*$ mark the ground and excited state, respectively. Note that we write the particle distribution function as the product $f(t,\mathbf{x},\mathbf{v}) = n(t,\mathbf{x})\tilde{f}(t,\mathbf{v})$, with functions depending on space and velocity. So one is able to describe both states in the same equation. In equation (68) $g^{\bullet}$ and g^* are statistical weights of the respective states. These weights represent the number of degenerated energy levels, i.e. how many atomic states with exact that energy exist. This is a Boltzmann type collision operator (cf. equation (36)), where the specific collision probabilities are realized with the respective cross sections $\sigma^{\bullet}$ and σ^* (cf. section 1.3). With v^+ defined in equation (55), the preceding factor of $\frac{v^+}{v}$ corrects for particle conservation.

This collision term simply accounts for all electrons that are lost and gained at a specific velocity due to collisions with particles from another species. In the process the targets are excited. Note that the second part of equation (68) includes collisions with excited particles, which de-excite and give the energy back to the electrons.

3.2.1 A simple inelastic collision term

An example of a simple inelastic collision term can be derived from equation (68) as

$$C_0(t,\mathbf{x},v) = \frac{v^+}{v}\nu(v^+)f(t,\mathbf{x},v^+) - \nu(v)f(t,\mathbf{x},v)\,. \tag{69}$$

This collision term model is very similar to equation (68). The difference is the missing second part, which describes the de-excitating collisions. One may neglect that term, if there is only a relatively small number of excited particles, e.g. if the specific state has a short half-life and the decay process does not have influence on the particle numbers.

3.2.2 Example: A very simple implementation

In the following we demonstrate how such a simple term may be used. To do so, we use a $0+1$ coordinate system, and therefore only investigate the velocity distribution of the given particles. In this setting we use the isotropic collision operator given in equation (69) in combination with a BGK-type collision operator, which is defined in equation (33).

The inelastic collision frequency used in connection to equation (69) is derived with equation (16) from a cross section given in Janev, Reiter, and Samm 2004. Thus, with the Heaviside step function

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{70}$$

the inelastic collision frequency follows as

$$\nu_{in}(v) = \nu_{in0}\frac{v_T}{v}\left(\beta_\sigma \ln\left(\frac{v^2}{v_{th}^2}\right) + \gamma_\sigma\left(1 - \frac{v_{th}^2}{v^2}\right)\right)\Theta(v - v_{th}) \tag{71a}$$

with

$$\nu_{in\,0} = \left(\frac{2e}{m}\right)^2 \frac{n_{in}\alpha_\sigma}{v_{th}^2 v_T}\,. \tag{71b}$$

In this example, the parameters are set to the values given in table 1.

e	$1.602{\cdot}10^{-19}\,\mathrm{C}$	T	$300\,\mathrm{K}$	α_σ	$10^{-24}\,\mathrm{m^2eV^2}$
m	$9.109{\cdot}10^{-31}\,\mathrm{kg}$	v_T	$95{\cdot}10^3\,\mathrm{m\,s^{-1}}$	β_σ	0.185
n_{in}	$10^{18}\,\mathrm{m^{-3}}$	W_{th}	$350\,\mathrm{K}$	γ_σ	-0.03
n_e	$10^{18}\,\mathrm{m^{-3}}$	v_{th}	$102.6{\cdot}10^3\,\mathrm{m\,s^{-1}}$		

Table 1: Values used for the parameters in this concrete example.

In this example, we do not assume any advection and observe only a single point. Furthermore, there is no external force. Even though we want a very simple example, having only a inelastic collision operator would not be realistic. Hence, we also include a BGK-type operator that thermalizes the medium and therefore works against the inelastic collision operator. Then we can compute the resulting equilibrium.

For the BGK-type operator we use equation (33) with the Maxwellian distribution for the equilibrium

$$f_{eq}(t,x,v) = n_e \frac{1}{\sqrt{\pi v_T^2}} e^{-v^2/v_T^2} \tag{72}$$

and a constant $\tau = 0.1\mathrm{s}$.

All this results in

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{v^+}{v}\nu_{in}(v^+)f(v^+) - \nu_{in}(v)f(v) + \nu_{BGK}(f_{eq}(t,x,v) - f(t,x,v))\,, \tag{73}$$

for which we want to find the equilibrium, i.e. $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$.

To solve the given equation, a simple fixed-point iteration was used. The resulting velocity distribution function is displayed in figure 4. The results are expected: due to the inelastic collisions, the electrons lose kinetic energy above the threshold. Thus, they are depleted above and collected below the threshold.

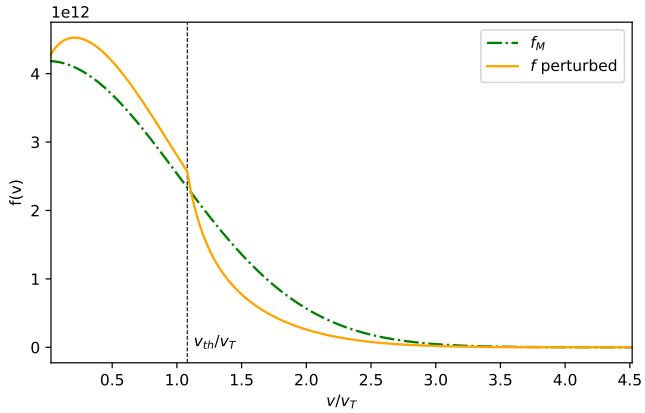

Figure 4: The phase space velocity distribution of electrons that collide inelastically in orange when compared to the undisturbed Maxwellian distribution in green. To counter the inelastic events, we use a BGK-type collision term to rethermalize the particles. The solid orange line shows the steady state.

4 Elastic Coulomb collisions

In this section we look at collision terms that are very important in plasma physics. Namely, those between charged particles, where the Coulomb force occurs. Therefore, in the following we show how to derive the collision operator corresponding to the Rutherford scattering, which is the relevant effect for collisions under a central Coulomb force.

4.1 Derivation of the Rosenbluth potentials

This is a summary of the work published in Rosenbluth, MacDonald, and Judd 1957. The collision operator is derived from the Fokker–Planck equation. To compute the averages in equation (48) the Rutherford scattering

$$\varsigma(\vartheta) = \frac{\gamma^2}{4u^4} \frac{1}{\sin^4(\frac{\vartheta}{2})} \tag{74}$$

is employed with

$$\gamma = \frac{e^2}{4\pi\epsilon_0 \cdot m_{\alpha\beta}} \,. \tag{75}$$

There e is the elementary charge, ϵ_0 is the vacuum permittivity, u is the relative speed, ϑ is the scattering angle and

$$m_{\alpha\beta} = \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \tag{76}$$

is the reduced mass.

From the center of mass (COM) system

$$(m_\alpha + m_\beta)\mathbf{V} = m_\alpha \mathbf{v}_\alpha + m_\beta \mathbf{v}_\beta \tag{77}$$

and its system velocity $\mathbf{V}$ and relative velocity $\mathbf{u} = \mathbf{v}_\beta - \mathbf{v}_\alpha$ (cf. with equation (4a)) we obtain

$$\mathbf{v}_\alpha = \mathbf{V} + \frac{m_\beta}{m_\alpha + m_\beta}\mathbf{u} \,. \tag{78}$$

Thus, we define equivalently to equation (3) the relative velocity change

$$\Delta\mathbf{u} = \mathbf{u}' - \mathbf{u} \tag{79}$$

as the velocity differences $\mathbf{u}$ (before) and $\mathbf{u}'$ (after a collision). This translates analogously equation (78) to the laboratory system as

$$\Delta\mathbf{v}_\alpha = \mathbf{v}'_\alpha - \mathbf{v}_\alpha = \left(\mathbf{V} + \frac{m_\beta}{m_\alpha + m_\beta}\mathbf{u}'\right) - \left(\mathbf{V} + \frac{m_\beta}{m_\alpha + m_\beta}\mathbf{u}\right) = \frac{m_\beta}{m_\alpha + m_\beta}\Delta\mathbf{u} \,. \tag{80}$$

In contrast to the general coordinate system given by $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$, an additional local orthonormal coordinate system L

$$\mathbf{e}^1_L = \frac{\mathbf{u}}{u} \,, \tag{81a}$$

$$\mathbf{e}^2_L = \frac{\mathbf{e}^3 \times \mathbf{u}}{\sqrt{(u^1)^2 + (u^2)^2}} \,, \tag{81b}$$

and

$$\mathbf{e}^3_L = \mathbf{e}^1_L \times \mathbf{e}^2_L \tag{81c}$$

is introduced with the relative speed u between the particles.

So, due to the elastic scattering, there is no change in relative speed, but only a deflection. The COM system itself stays unperturbed anyhow. Thus, the changes in relative velocity depend on the deflection angle ϑ and the angle φ of the corresponding plane spanned by $\mathbf{u}_L$ and $\mathbf{u}'_L$ as follows

$$\Delta u_L^1 = u\,(\cos(\vartheta) - 1) = -2u\,\sin^2(\tfrac{\vartheta}{2})\,, \tag{82a}$$

$$\Delta u_L^2 = u\,\sin(\vartheta)\cos(\varphi) = 2u\,\cos(\tfrac{\vartheta}{2})\sin(\tfrac{\vartheta}{2})\cos(\varphi)\,, \tag{82b}$$

and

$$\Delta u_L^3 = u\,\sin(\vartheta)\cos(\varphi) = 2u\,\cos(\tfrac{\vartheta}{2})\sin(\tfrac{\vartheta}{2})\sin(\varphi)\,, \tag{82c}$$

which is illustrated in figure 5.

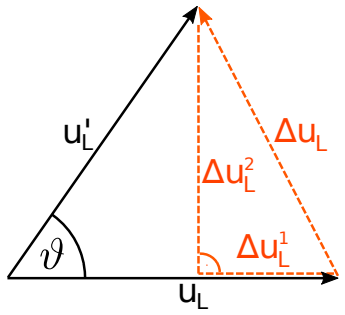

Figure 5: An elastic collision does not change the absolute relative velocity, only its direction. Thus one can simply view the velocity change depending on the scattering angle ϑ. So, in this graphic of a scattering event in the 1-2 plane, one can easily find $\Delta\mathbf{u}_L$ and its components in orange.

At this point we can already compute the angular integrals of equation (48). When integrating there might be complications due to divergence at $\vartheta \to 0$. This is alleviated by introducing a minimum angle $\vartheta_{min} > 0$, which corresponds to the lowest angle at which a deflection is still recognized as such. In the following we detail the computation of the first component as an example. Todo so we define the angular mean and variance as

$$\{\Delta u_L^\mu\} \equiv \int_{S^2} \varsigma\; u\Delta u_L^\mu \,\mathrm{d}\Omega \tag{83a}$$

and

$$\{\Delta u_L^\mu \Delta u_L^\nu\} \equiv \int_{S^2} \varsigma\; u\Delta u_L^\mu \Delta u_L^\nu \,\mathrm{d}\Omega\,. \tag{83b}$$

$$\begin{aligned}
\{\Delta u_L^1\} &= \int_{\vartheta_{min}}^{\pi}\int_0^{2\pi} \varsigma\; u\Delta u_L^1 \sin(\vartheta)\mathrm{d}\varphi\mathrm{d}\vartheta = \int_{\vartheta_{min}}^{\pi}\int_0^{2\pi} \left(\frac{\gamma^2}{4u^4}\frac{1}{\sin^4(\frac{\vartheta}{2})}\right) u\left(-2u\,\sin^2(\tfrac{\vartheta}{2})\right)\,\sin(\vartheta)\mathrm{d}\varphi\mathrm{d}\vartheta \\
&= -\frac{\gamma^2}{u^2}\int_{\vartheta_{min}}^{\pi}\int_0^{2\pi} \frac{\cos(\frac{\vartheta}{2})}{\sin(\frac{\vartheta}{2})}\mathrm{d}\varphi\mathrm{d}\vartheta = -2\pi\frac{\gamma^2}{u^2}\int_{\vartheta_{min}}^{\pi} \frac{\cos(\frac{\vartheta}{2})}{\sin(\frac{\vartheta}{2})}\mathrm{d}\vartheta \\
&= -4\pi\frac{\gamma^2}{u^2}\ln\left(\sin(\tfrac{\vartheta}{2})\right)\Big|_{\vartheta_{min}}^{\pi} = -4\pi\frac{\gamma^2}{u^2}\left(\ln\left(\sin(\tfrac{\pi}{2})\right) - \ln\left(\sin(\tfrac{\vartheta_{min}}{2})\right)\right) \\
&= 4\pi\frac{\gamma^2}{u^2}\ln\left(\sin(\tfrac{\vartheta_{min}}{2})\right) \approx 4\pi\frac{\gamma^2}{u^2}\ln\left(\tfrac{\vartheta_{min}}{2}\right) = -4\pi\frac{\gamma^2}{u^2}\ln\left(\frac{2}{\vartheta_{min}}\right) = -4\pi\frac{\gamma^2}{u^2}L\,,
\end{aligned} \tag{84}$$

	$\mathbf{e}^1$	$\mathbf{e}^2$	$\mathbf{e}^3$
$\mathbf{e}_L^1$	$\frac{u^1}{u}$	$\frac{u^2}{u}$	$\frac{u^3}{u}$
$\mathbf{e}_L^2$	$\frac{-u^2}{\sqrt{(u^1)^2+(u^2)^2}}$	$\frac{u^1}{\sqrt{(u^1)^2+(u^2)^2}}$	0
$\mathbf{e}_L^3$	$\frac{-u^1u^3}{u\sqrt{(u^1)^2+(u^2)^2}}$	$\frac{-u^2u^3}{u\sqrt{(u^1)^2+(u^2)^2}}$	$\frac{(u^1)^2+(u^2)^2}{u\sqrt{(u^1)^2+(u^2)^2}}$

Table 2: The scalar products of all the possible vector combinations.

where we introduced the Landau number

$$L = \ln(2/\vartheta_{min})\,. \tag{85}$$

This number basically represents the minimum cut-off angle, below which we do not count a particle interaction as a collision anymore.

For this angular mean the other components are zero, hence we summarize

$$\{\Delta\mathbf{u}_L\} = \begin{pmatrix}\{\Delta u_L^1\}\\ \{\Delta u_L^2\}\\ \{\Delta u_L^3\}\end{pmatrix} = \begin{pmatrix}-4\pi\frac{\gamma^2}{u^2}L\\ 0\\ 0\end{pmatrix} \tag{86a}$$

and correspondingly[2]

$$\{\Delta\mathbf{u}_L\Delta\mathbf{u}_L^T\} = \begin{pmatrix}\{\Delta u_L^1\Delta u_L^1\} & \{\Delta u_L^1\Delta u_L^2\} & \{\Delta u_L^1\Delta u_L^3\}\\ \{\Delta u_L^2\Delta u_L^1\} & \{\Delta u_L^2\Delta u_L^2\} & \{\Delta u_L^2\Delta u_L^3\}\\ \{\Delta u_L^3\Delta u_L^1\} & \{\Delta u_L^3\Delta u_L^2\} & \{\Delta u_L^3\Delta u_L^3\}\end{pmatrix} \approx \begin{pmatrix}0 & 0 & 0\\ 0 & 4\pi\frac{\gamma^2}{u}L & 0\\ 0 & 0 & 4\pi\frac{\gamma^2}{u}L\end{pmatrix}. \tag{86b}$$

Therein terms that are much smaller than L have been neglected. Anyhow, many of the entries are zero.

When transforming these results into the coordinate system from which we started, we get

$$\Delta u^\mu = ((\mathrm{e}^\mu)_\eta(\mathrm{e}_L{}^\sigma)_\eta)\Delta u_L^\sigma \tag{87a}$$

and

$$\Delta u^\mu\Delta u^\nu = ((\mathrm{e}^\mu)_\eta(\mathrm{e}_L{}^\iota)_\eta)((\mathrm{e}^\nu)_\omega(\mathrm{e}_L{}^\sigma)_\omega)\Delta u_L^\iota\Delta u_L^\sigma\,. \tag{87b}$$

Therefore, the following scalar products, which are detailed in table 2, are used to transform back into the laboratory system. Not surprisingly, the given transformation matrix has determinant 1. Thus, we conclude with the transformation from the COM system to the general system (cf. equation (80)) the following terms:

$$\{\Delta v^\mu\}_\alpha = -\Gamma_\alpha\frac{m_\alpha}{m_{\alpha\beta}u^2}\frac{u^\mu}{u} \tag{88a}$$

$$\{\Delta v^\mu\Delta v^\nu\}_\alpha = \Gamma_\alpha\frac{1}{u}\left(\delta^{\mu\nu}-\frac{u^\mu u^\nu}{u^2}\right), \tag{88b}$$

where

$$\Gamma_\alpha = 4\pi\gamma^2L\left(\frac{m_\beta}{m_\alpha+m_\beta}\right)^2 = \frac{e^4L}{4\pi\epsilon_0^2m_\alpha^2} \tag{89}$$

[2]Note that there is a mistake in Rosenbluth, MacDonald, and Judd 1957 concerning equation (86b): there is u^2 instead of u, which consequently is carried forward for the following paragraph, but the corresponding error is corrected afterwards.

has been introduced.

Equation (88a) is obtained relatively effortless, because the mean velocity change in the COM system has only one non-zero element. In contrast, the computation of the double scalar product of equation (87b) is only simplified a little-bit by the fact that equation (88b) has only non-zero elements in $\{(2,2),(3,3)\}$. Thus, we can compute for $\eta,\sigma \in \{2,3\}$

$$(\mathrm{e}_\eta^\mu \mathrm{e}_{L\iota}^{\ \eta})(\mathrm{e}_\omega^\mu \mathrm{e}_{L\sigma}^{\ \omega})\{\Delta u_L^\iota \Delta u_L^\sigma\} = \begin{pmatrix} \frac{(u^2)^2+(u^3)^2}{\|\mathbf{u}\|^2} & -\frac{u^1u^2}{\|\mathbf{u}\|^2} & -\frac{u^1u^3}{\|\mathbf{u}\|^2} \\ -\frac{u^1u^2}{\|\mathbf{u}\|^2} & \frac{(u^1)^2+(u^3)^2}{\|\mathbf{u}\|^2} & -\frac{u^2u^3}{\|\mathbf{u}\|^2} \\ -\frac{u^1u^3}{\|\mathbf{u}\|^2} & -\frac{u^2u^3}{\|\mathbf{u}\|^2} & \frac{(u^1)^2+(u^2)^2}{\|\mathbf{u}\|^2} \end{pmatrix} = \delta^{\mu\nu} - \frac{u^\mu u^\nu}{\|\mathbf{u}\|^2}\,. \quad (90)$$

There one can observe that 0 is an eigenvalue with eigenwert u. Hence, the given matrix has a determinant of 0.

Consequently, one can write equation (88) with ($\|\mathbf{u}\| = \|\mathbf{v}_\alpha - \mathbf{v}_\beta\|$) as

$$\{\Delta v^\mu\}_{\,\alpha} = \Gamma_\alpha \frac{m_\alpha}{m_{\alpha\beta}} \frac{\partial}{\partial v_\alpha^\mu} \frac{1}{\|\mathbf{u}\|} \quad (91a)$$

and

$$\{\Delta v^\mu \Delta v^\nu\}_{\,\alpha} = \Gamma_\alpha \frac{\partial^2}{\partial v^\mu \partial v^\nu} \|\mathbf{u}\|\,, \quad (91b)$$

which allows us to formulate the mean velocity changes in first and second order as

$$\boxed{\langle \Delta v^\mu \rangle_\alpha = \Gamma_\alpha \frac{\partial h_\alpha}{\partial v^\mu}} = \sum_\beta \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f_\beta(t,\mathbf{x},\mathbf{v}_\beta)\{\Delta v^\mu\}_\alpha \ \mathrm{d}\mathbf{v}_\beta \quad (92a)$$

and

$$\boxed{\langle \Delta v^\mu \Delta v^\nu \rangle_\alpha = \Gamma_\alpha \left(\frac{\partial^2 g}{\partial v^\mu \partial v^\nu} \right)} = \sum_\beta \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f_\beta(t,\mathbf{x},\mathbf{v}_\beta)\{\Delta v^\mu \Delta v^\nu\}_\alpha \ \mathrm{d}\mathbf{v}_\beta\,. \quad (92b)$$

There the *Rosenbluth potentials*

$$\boxed{h_\alpha(t,\mathbf{x},\mathbf{v}_\alpha) = \sum_\beta \frac{m_\alpha + m_\beta}{m_\beta} \int\limits_{\mathbb{R}^3_{\mathbf{v}}} \frac{f_\beta(t,\mathbf{x},\mathbf{v}_\beta)}{\|\mathbf{v}_\alpha - \mathbf{v}_\beta\|} \ \mathrm{d}\mathbf{v}_\beta} \quad (93a)$$

and

$$\boxed{g_\alpha(t,\mathbf{x},\mathbf{v}_\alpha) = \sum_\beta \int\limits_{\mathbb{R}^3_{\mathbf{v}}} f_\beta(t,\mathbf{x},\mathbf{v}_\beta)\|\mathbf{v}_\alpha - \mathbf{v}_\beta\| \ \mathrm{d}\mathbf{v}_\beta} \quad (93b)$$

have been introduced.

As potentials in $\mathbf{v}$, g_α and h_α satisfy *Poisson's equation* like

$$\frac{\partial^2 g_\alpha}{\partial (v^\mu)^2} = 2h_\alpha \quad (94a)$$

and

$$\frac{\partial^2 h_\alpha}{\partial (v^\mu)^2} = f_\alpha\,. \quad (94b)$$

We continue the current reasoning and write the *Fokker–Planck equation* (44)

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = -\frac{\partial}{\partial v^\mu}\left(f_\alpha \cdot \langle \Delta v^\mu \rangle\right) + \frac{1}{2}\frac{\partial^2}{\partial v^\mu \partial v^\nu}\left(f_\alpha \cdot \langle \Delta v^\mu \Delta v^\nu \rangle\right)$$

with *Rosenbluth potentials* as

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = -\Gamma_\alpha \frac{\partial}{\partial v^\mu}\left(f\frac{\partial h_\alpha}{\partial v^\mu}\right) + \frac{1}{2}\Gamma_\alpha \frac{\partial^2}{\partial v^\mu \partial v^\nu}\left(f\frac{\partial^2 g_\alpha}{\partial v^\mu \partial v^\nu}\right). \tag{95}$$

Let us note here that it could be feasible to compute the potentials g and h numerically. This is especially true if f_β is not fixed and also a part of the simulation.

As suggested in Karney 1986 equation (95) can also be rewritten as

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = -\Gamma_\alpha \frac{\partial}{\partial v^\mu}\left(f\frac{\partial h_\alpha}{\partial v^\mu}\right) + \frac{1}{2}\Gamma_\alpha \left(\frac{\partial}{\partial v^\mu}\left(\left(\frac{\partial f}{\partial v^\nu}\right)\frac{\partial^2 g_\alpha}{\partial v^\mu \partial v^\nu}\right) + \frac{\partial}{\partial v^\mu}\left(f\frac{\partial}{\partial v^\mu}\frac{\partial^2 g_\alpha}{\partial (v^\nu)^2}\right)\right), \tag{96}$$

which can be continued with equation (94a) like

$$\boxed{\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = \frac{1}{2}\Gamma_\alpha \left(\frac{\partial}{\partial v^\mu}\left(\left(\frac{\partial f}{\partial v^\nu}\right)\frac{\partial^2 g_\beta}{\partial v^\mu \partial v^\nu}\right)\right)}. \tag{97}$$

It can also be written as

$$\boxed{\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = \frac{\Gamma_\alpha}{2}\nabla \cdot ((\nabla\nabla g_\alpha)\nabla f)} \tag{98}$$

as a shorthand notation. There, the tensor $(\nabla\nabla)^{\mu\nu} = \frac{\partial^2}{\partial v^\mu \partial v^\nu}$ is used.

If we look at equation (98) and compare it to the definition of the collision fluxes given in section 2, one can read the corresponding flux field straightforward as

$$\mathbf{S} = -\frac{\Gamma_\beta}{2}(\nabla\nabla g_\beta)\nabla f\,. \tag{99}$$

4.2 Example of stationary ions

At this point we can already analyze an interesting example: The case of electrons (e) colliding with massive (quasi-)stationary ions (i). There the assumptions

1. $m_i \gg m_e$
2. $\|\mathbf{v}_e\| \gg \|\mathbf{v}_i\| \approx 0$

are valid and allow us to define the ion distribution with the particle density n_i as

$$f_i(t, \mathbf{x}, \mathbf{v}) = n_i \delta(\mathbf{v}) \tag{100}$$

and obtain

$$h_e(t, \mathbf{x}, \mathbf{v}) = \frac{m_e + m_i}{m_i}\frac{n_i}{\|\mathbf{v}\|} \approx \frac{n_i}{\|\mathbf{v}\|} \tag{101a}$$

and

$$g_e(t, \mathbf{x}, \mathbf{v}) = n_i \|\mathbf{v}\|\,, \tag{101b}$$

with the *Dirac delta distribution* δ.

When differentiating these *Rosenbluth potentials* as described in equation (91) we get

$$\langle \Delta v_\mu \rangle_e = \Gamma_e n_i \frac{\partial}{\partial v_\mu} \frac{1}{v} = -\Gamma_e n_i \frac{v^\mu}{v^3} \tag{102a}$$

$$\langle \Delta v^\mu \Delta v^\nu \rangle_e = \Gamma_e n_i \frac{\partial^2}{\partial v^\nu \partial v^\mu} v = \Gamma_e n_i \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right), \tag{102b}$$

which we can enter into the already stated *Fokker–Planck equation* (44)

$$\frac{df}{dt} = -\frac{\partial}{\partial v^\mu} \left(f \langle \Delta v^\mu \rangle \right) + \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial^\nu} \left(f \langle \Delta v^\mu \Delta v^\nu \rangle \right)$$

to find

$$\begin{aligned} \frac{1}{\Gamma_e n_i} \frac{df}{dt} &= \frac{\partial}{\partial v^\mu} \left(f \frac{\partial}{\partial v^\mu} \frac{1}{v} \right) + \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial^\nu} \left(f \left(\frac{\partial^2}{\partial v^\nu \partial v^\mu} v \right) \right) \\ &= \frac{\partial}{\partial v^\mu} \left(f \frac{v^\mu}{v^3} \right) + \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial^\nu} \left(f \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) \right). \end{aligned} \tag{103}$$

As a next step let us take a detailed look at the second term

$$\begin{aligned} \frac{\partial}{\partial^\nu} \left(f \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) \right) &= \frac{\partial f}{\partial v^\nu} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) + f \left(-\frac{\delta^{\mu\nu} v^\nu}{v^3} - \frac{\delta^{\mu\nu} v^\nu}{v^3} - \frac{v^\mu}{v^3} + 3 \frac{v^\mu v^{\nu 2}}{v^5} \right) \\ &= \frac{\partial f}{\partial^\nu} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) + f \left(-\frac{v^\mu}{v^3} - \frac{v^\mu}{v^3} - \frac{3 v^\mu}{v^3} + 3 \frac{v^\mu v^{\nu 2}}{v^5} \right) \\ &= \frac{\partial f}{\partial^\nu} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) + f \left(-2 \frac{v^\mu}{v^3} \right), \end{aligned} \tag{104}$$

where in the second part the sum over ν has already been executed. Thus, we insert equation (104) into equation (103) to obtain

$$\begin{aligned} \frac{1}{\Gamma_e n_i} \frac{df}{dt} &= \frac{\partial}{\partial v^\mu} \left(f \frac{v^\mu}{v^3} \right) + \frac{1}{2} \frac{\partial}{\partial v^\mu} \left(\frac{\partial f}{\partial^\nu} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) + f \left(-2 \frac{v^\mu}{v^3} \right) \right) \\ &= \cancel{\frac{\partial}{\partial v^\mu} \left(f \frac{v^\mu}{v^3} - f \frac{v^\mu}{v^3} \right)} + \frac{1}{2} \frac{\partial}{\partial v^\mu} \left(\frac{\partial f}{\partial^\nu} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) \right) = \frac{1}{2} \frac{\partial}{\partial v^\mu} \left(\left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) \frac{\partial f}{\partial^\nu} \right). \end{aligned} \tag{105}$$

So we can write finally with

$$A^{\mu\nu} = \frac{1}{2} \left(\frac{\delta^{\mu\nu}}{v} - \frac{v^\mu v^\nu}{v^3} \right) \tag{106}$$

we write

$$\frac{1}{\Gamma_e n_i} \frac{df}{dt} = \nabla \cdot (A \nabla f). \tag{107}$$

4.3 Interspecies collisions

In this section we look at another case. The case, where identical particles collide with each other. It is especially relevant when thinking of electron-electron collisions. They are very important for simulating plasmas, due to their thermalizing effects and their sheer number in most plasma settings of interest.

So for identical particles we can not use the same simplifications as in section 4.2. All we can presume is that

$$m_\alpha = m_\beta \,, \tag{108}$$

even though we also know that

$$f_\alpha = f_\beta \,. \tag{109}$$

This leads to the following *Rosenbluth potentials* (equation (93)) of

$$h(t, \mathbf{x}, \mathbf{v}) = 2 \int\limits_{\mathbb{R}^3_\mathbf{v}} \frac{f(t, \mathbf{x}, \tilde{\mathbf{v}})}{\|\mathbf{v} - \tilde{\mathbf{v}}\|} \, \mathrm{d}\tilde{\mathbf{v}} \tag{110a}$$

and

$$g(t, \mathbf{x}, \mathbf{v}) = \int\limits_{\mathbb{R}^3_\mathbf{v}} f(t, \mathbf{x}, \tilde{\mathbf{v}}) \|\mathbf{v} - \tilde{\mathbf{v}}\| \, \mathrm{d}\tilde{\mathbf{v}} \,. \tag{110b}$$

With equation (95) these expressions lead to the corresponding collision term for identical particles (or for particles of the same mass right now). Thus, the general collision operator becomes a global operator because of the integral. It therefore requires many memory access, which is bad for the speed of computation.

Therefore, Karney 1986, p. 199 discusses several simplifications:

1. *Linearization* – With the linearized phase space density, the interspecies collision term follows as

$$C(f_0 + f_1, f_0 + f_1) = \cancelto{0}{C(f_0, f_0)} + C(f_0, f_1) + C(f_1, f_0) + \cancelto{0}{C(f_1, f_1)} \tag{111}$$

   As the thermal zero order terms in the linearization deliver 0 net number change, one needs only to compute the collision term of the first order perturbation with a thermalized background. Here, it is assumed that f_0 is the Maxwellian. And if both particle species are thermalized, the collision operator is zero. The last term in equation (111) is neglected because the linear perturbation term is be small.

2. *Drifting Maxwellian* – If there is a global velocity difference to the background medium, one can simply apply a translation in the velocity space.

Furthermore, all the approximation cases mentioned in section 2.4 do apply. But naturally it strongly depends on situation and numerical approach whether any simplification is advantageous.

5 The Landau collision operator

Villani 2002, p. 21, Chuyanov 2011, Balescu 1975 give the Landau collision operator as

$$C_L(f_\alpha, f_\beta) = \nabla_{\mathbf{v}_\alpha} \cdot \int\limits_{\mathbb{R}^3_\mathbf{v}} a(\mathbf{v}_\alpha - \mathbf{v}_\beta)(f_\beta(\nabla_v f)_\alpha - f_\alpha(\nabla_v f)_\beta) \, \mathrm{d}\mathbf{v}_\beta \,, \tag{112}$$

where the matrix $a(\mathbf{u})$ is defined as

$$a^{\mu\nu}(\mathbf{u}) = \frac{L}{\|\mathbf{u}\|} \left(\delta^{\mu\nu} - \frac{u^\mu u^\nu}{\|\mathbf{u}\|^2} \right) . \tag{113}$$

L is a constant and δ is the Kronecker delta.

It is eye-catching that this term is equivalent to equation (106). This is due to the fact that according to Chuyanov 2011 the Landau collision operators becomes the linear Fokker–Planck equation for rarefied test particles in an approximate equilibratory background medium.

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