Simple example

Mirko Residori

We are given the system of ordinary differential equations

$$\begin{cases} y_1'(t) = -3\, y_1(t), \\ y_2'(t) = -5\, y_2(t), \\ y_1(0) = 1, \\ y_2(0) = 2. \end{cases}$$

The analytical solution is given by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t} \\ 2\mathrm{e}^{-5t} \end{bmatrix}.$$

We solve the system with Euler's method. We take as final time $t = T$ and we choose different time discretizations, e.g., we use $N = 512$, 1024, 2048, 4096 steps. The step size will be

$$h = T/N$$

and the numerical solution given by Euler's method will be

$$\mathbf{y}^{n+1} = \mathbf{y}^n + hf(t_n, \mathbf{y}^n), \quad 0 \le n \le N$$

where

$$\mathbf{y}^{\mathbf{n}} = [y_1^n, y_2^n]^T, \quad f(t_n, \mathbf{y}^n) = [-3y_1^n, -5y_2^n]^T, \quad \mathbf{y}^0 = [1, 2]^T.$$

Each of the time discretizations will provide us a numerical solution

$$\mathbf{y}^N = \begin{bmatrix} y_1^N \\ y_2^N \end{bmatrix}.$$

In order to test the order of converge of the Euler's method (which is supposed to be 1) we compute the norm

$$\mathrm{err}_N = \left\| \begin{bmatrix} y_1^N \\ y_2^N \end{bmatrix} - \begin{bmatrix} \mathrm{e}^{-3T} \\ 2\mathrm{e}^{-5T} \end{bmatrix} \right\|_2.$$

For a more general problem, where we don't know the analytical solution, we can substitute the analytical solution by a reference solution. That means, we compute a solution $\mathbf{y}^{N_{\mathrm{ref}}}$ with $N_{\mathrm{ref}} \gg N$ and we evaluate the error

$$\mathrm{err}_N = \left\| \begin{bmatrix} y_1^N \\ y_2^N \end{bmatrix} - \begin{bmatrix} y_1^{N_{\mathrm{ref}}} \\ y_2^{N_{\mathrm{ref}}} \end{bmatrix} \right\|_2.$$

Notice that the error is computed at the final time only. In a double logaritmic plot we show the order of convergence as follows. On the x axis we plot the different Ns. On the y axis we plot the error for the different Ns. We finally plot a straight line of slope -1 and we see if it is parallel to the error line. If this is the case, then our numerical method converges with the right order. A simple python code to plot a straight line of slope $-k$ is given below.

```python
# Compute the reference solution y_ref.
# Compute the numerical solution y for different Ns.
# Compute the error between y_ref and y.
nrange = array([2**9,2**10,2**11,2**12])
# error plot
loglog(nrange,error,label="err")
# straight line of slope k
k = -1
loglog(nrange,nrange.astype(float)**(k),label="slope -1")
legend()
```