(4.1) Verify that the Burgers equation with initial data

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

has the weak solution

$$u(t,x) = \begin{cases} 1 & x \leq -t/2 \\ -2 & -t/2 \leq x \leq 0 \\ 2 & 0 \leq x \leq t \\ 0 & t \leq x \end{cases}$$

that consists of three shocks. Does this solution satisfy the Lax entropy condition?

(4.2) Show that the equations

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \qquad \text{and} \qquad (u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

have different weak solutions but the same classical solutions.

(4.3) Verify that the upwind flux

$$F(v,w) = f(v) \qquad \text{for } f' > 0$$

is consistent, i.e., it satisfies the two conditions given in the lecture.

(4.4) Find the unique weak solution of Burgers' equation with initial data

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

that satisfies the entropy condition.

(4.5) Compute the solution of Exercise 4.4 with the conservative upwind scheme.

(4.6) Solve Burger's equation

$$u_t + uu_x = 0, \qquad 0 \leq x \leq 1, \quad 0 \leq t \leq 0.5$$

on a computer with Riemann initial data

$$u_0(x) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x \geq 0 \end{cases}$$

for $\tau/h = 0.5$ and various values of h using

(a) the upwind scheme;

(b) the conservative upwind scheme.

Determine numerically the speed of the shock front (as a function of h) and compare it with the exact result.